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Three Attractive Osculating Walkers and a Polymer Collapse Transition

John W. Essam[1]

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Consider n interacting lock-step walkers in one dimension which start at the points $\{0, 2, 4,..., 2(n-1)\}$ and at each tick of a clock move unit distance to the left or right with the constraint that if two walkers land on the same site their next steps must be in the opposite direction so that crossing is avoided. When two walkers visit and then leave the same site an osculation is said to take place. The space-time paths of these walkers may be taken to represent the configurations of n fully directed polymer chains of length t embedded on a directed square lattice. If a weight λ is associated with each of the i osculations the partition function is $Z_t^{(n)}(\lambda) = \sum_{i=0}^{\lfloor \frac{(n-1)t}{2} \rfloor} z_{t,i}^{(n)} \lambda^i$ where $z_{t,i}^{(n)}$ is the number of t-step configurations having i osculations. When $\lambda = 0$ the partition function is asymptotically equal to the number of vicious walker star configurations for which an explicit formula is known. The asymptotics of such configurations was discussed by Fisher in his Boltzmann medal lecture. Also for $n = 2$ the partition function for arbitrary λ is easily obtained by Fisher's necklace method. For $n > 2$ and $\lambda \neq 0$ the only exact result so far is that of Guttmann and Vöge who obtained the generating function $G^{(n)}(\lambda, u) \equiv \sum_{t=0}^{\infty} Z_t^{(n)}(\lambda)\, u^t$ for $\lambda = 1$ and $n = 3$. The main result of this paper is to extend their result to arbitrary λ. By fitting computer generated data it is conjectured that $Z_t^{(3)}(\lambda)$ satisfies a third order inhomogeneous difference equation with constant coefficients which is used to obtain

$$G^{(3)}(\lambda, u) = \frac{(\lambda-3)(\lambda+2) - \lambda\,(12 - 5\,\lambda + \lambda^2)\,u - 2\lambda^3\,u^2 + 2(\lambda-4)(\lambda^2\,u^2 - 1)\,c(2u)}{(\lambda - 2 - \lambda^2\,u)\,(\lambda - 1 - 4\lambda u - 4\lambda^2\,u^2)}$$

where $c(u) = \frac{1-\sqrt{1-4u}}{2u}$, the generating function for Catalan numbers. The nature of the collapse transition which occurs at $\lambda = 4$ is discussed and extensions to

[1] Department of Mathematics, Royal Holloway, University of London, Surrey TW20 0EX, England; e-mail: j.essam@rhul.ac.uk

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higher values of n are considered. It is argued that the position of the collapse transition is independent of n.

KEY WORDS: Osculating walkers; polymer networks; critical exponents; collapse transition.

1. INTRODUCTION AND SUMMARY

The problem of interacting random walks was considered by Fisher[1] in his Boltzmann Medal Lecture where many applications in statistical physics and chemistry were discussed. He introduced the term "vicious walkers" to describe the behaviour of a set drunks who each perform a random walk on a one dimensional lattice and if any two arrive on the same site they shoot one another. More specifically he analysed the asymptotic behaviour as $t \to \infty$ of the probability $P_t^{(n)}$ that walkers starting at positions $\{0, 2, \ldots, 2(n-1)\}$ survive for at least t steps and found that

$$P_t^{(n)} \sim 1/t^{\frac{1}{4}n(n-1)}. \tag{1}$$

A similar formula was given for the probability that the walkers survive and reunite anywhere after t steps at spacing two apart but a different n dependence was found for the exponent. These critical exponents were found to be relevant to the physical applications considered. Later Forrester[2] extended Fisher's work to walkers near a cliff so that not only did the walkers have to avoid one another but had to avoid stepping off the cliff. The survival and reunion probabilities were found to have modified critical exponents.

The space-time trajectories of vicious walkers may also be considered as fully directed polymer chains embedded on a square lattice and which are not allowed to touch one another (see Fig. 1 but with intersections avoided). The polymer configurations which contribute to the survival and reunion probabilities were called by Duplantier[3] "stars" and "watermelons" respectively. He found critical exponent formulae for general polymer networks embedded on an undirected square lattice and the corresponding formulae for directed networks, with and without a surface, were found by Zhao *et al.*[4]

Fisher's analysis[1, 5] used the method of images to express the number of n-walk configurations as an $n \times n$ determinant the elements of which were the numbers of single walk configurations. He then substituted an asymptotic form for each element and was able to evaluate the determinant and hence obtain the asymptotic form for any number of walkers. Later Arrowsmith *et al.*[6] observed that walk configurations correspond to

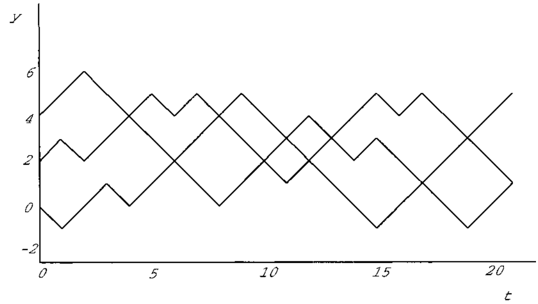

Fig. 1. Space-time paths of three osculating walkers. Each walker has made 21 steps and there are 9 osculations.

integer directed flows which were shown to have polynomial dependence on n and explicit polynomial formulae for the total number of star configurations and the number of watermelon configurations with an arbitrary fixed endpoint were conjectured. Their results were based on the examination of flow configurations for small values of t. Essam and Guttmann[7] were able to evaluate Fisher's determinant (also known to combinatorics community as the Gessel–Viennot determinant[8]) and thereby prove the polynomial formulae for stars and watermelons with fixed endpoints. Also simple recurrence relations were found for the total number of stars which supported the formula of Arrowsmith *et al.* Exact results for the numbers of watermelons with free endpoints are more difficult to obtain but recurrence relations were proven for each $n \leqslant 6$. The order of the recurrence was $\lfloor \frac{1}{2}(n+1) \rfloor$ and the coefficients, which were polynomials in t, had degrees which increased with n in an unpredictable manner. Critical exponents obtained from these formulae and recurrence relations were in agreement with Fisher's asymptotic analysis.

An alternative approach to the enumeration of vicious walker configurations was used by Guttmann *et al.*[9] who found a mapping to Young tableaux. Using known results in the theory of tableaux they were able to prove both the fixed endpoint formulae and also the formula for the total number of stars. The latter was known in tableaux theory as the Bender–Knuth conjecture[10] of which there are now several proofs all of which are very lengthy. Krattenhaler *et al.*[11] found corresponding formulae for polymers in the presence of a wall by mapping the configurations to simplectic tableaux. The Gessel–Viennot formulae for fixed endpoint configurations in the presence of a wall had previously been evaluated by Brak and Essam[12] who also included the number of wall contacts as a parameter. Recently a much neater proof of the free star formula without a wall has been given by Nagao and Forrester[13] using a method from random matrix theory.[14] They were also able to find the k-point correlation functions. A connection between vicious walkers and random matrix theory has also been discussed by Baik.[15]

In this paper the walkers are not vicious. When two walkers arrive at the same site, on the next step they are not allowed to pass and must move away from one another on to the sites from which they arrived. In the corresponding polymer chain picture the chains can intersect but must have no bonds in common. An intersection other than on the last step is said to be an "osculation." Figure 1 shows the space-time trajectories of three osculating walkers with 21 steps and 9 osculations which may also be considered as a polymer network. It is also a configuration of the 6-vertex model whereas networks obtained from vicious walkers have only 5 distinct types of vertex.

The application considered here is to a polymer collapse transition. Such a collapse may be induced by including a Boltzmann weight λ with each osculation. Suppose the initial points of the n-chains are at $\{(0,0), (0,2),...(0, 2(n-1)\}$ and let $z_{t,i}^{(n)}$ be the total number of configurations having i osculations, then the partition function is defined by

$$Z_t^{(n)}(\lambda) \equiv \sum_{i=0}^{\left\lfloor \frac{(n-1)t}{2} \right\rfloor} z_{t,i}^{(n)} \lambda^i \tag{2}$$

The Boltzmann factor λ^i makes the chains mutually repulsive or attractive depending on whether $\lambda < 1$ or $\lambda > 1$. The repulsive case is less interesting since it is qualitatively similar to the non-intersecting chain network based on "vicious walkers;" in the case $\lambda = 0$ the partition function $Z_t^{(n)}(0)$ is the same, asymptotically, as the total number vicious walker star configurations. This similarity persists into the attractive region as far as the collapse transition point λ_c at which the chains begin to stick together. It will be argued that $\lambda_c = 4$ for any number of chains. The grand partition function

$$G^{(n)}(\lambda, u) \equiv \sum_{t=0}^{\infty} Z_t^{(n)}(\lambda)\, u^t = \sum_{t=0}^{\infty} \sum_{i=0}^{\left\lfloor \frac{(n-1)t}{2} \right\rfloor} z_{t,i}^{(n)} \lambda^i u^t \tag{3}$$

will also be needed in studying the transition.

Another reason for studying the osculating walk model is that it is similar to, but simpler than, the following interacting walk model related to directed percolation. Arrowsmith *et al.*[6] showed that the number of integer flows between a source u and sink v of strength n on a directed percolation cluster has polynomial dependence on n. They also showed that the expected number of such flows $E(f_{uv}(n))$ in the limit $n \to 0$ gives the pair connectedness. Finally it was observed that $E(f_{uv}(n))$ for a source and sink separated by t-steps can be written as a sum over vicious walker watermelon configurations with an overall weight $(p_s p_b)^{nt}$ and a weight $1/p_s$ for each close encounter (i.e., each time two walkers come within distance two of each other) and a weight $1/p_b$ for each pair of successive close encounters. $\sum_v E(f_{uv}(n))$ is like the grand partition above with $u=(p_s p_b)^n$ and λ^i replaced by the factors $1/p_s$ and $1/p_b$. The limit $n \to 0$ of the grand partition function gives the expected cluster size.

The results of this paper for the osculating walk model are mainly restricted to $n=2$ and 3 except at the collapse transition it is argued that $Z_t^{(n)}(4)=2^{nt}$. The reason for this limitation is that the determinant formula[1, 8] for the number of vicious walkers no longer applies since the weights depend on the relative positions of the walkers and not just on the bonds traversed. Brak[16] has conjectured a general formula for the generating function of n osculating walks for general λ but with fixed endpoints. This replaces the vicious walker determinant by a sum over permutations. The formula also involves multiple summations and a further summation over endpoints to obtain the formula for three walkers given here has not been possible. The partition function for two walkers is easily found in various ways[1, 17, 18] (for example, see Section 2) but for $n=3$ we have resorted to computer enumeration. However the partition function found is conjectured to be exact although no proof has been found.

Recently Guttmann and Vöge[17] found an explicit formula for $G^{(3)}(1, u)$, that is in the case when all osculating configurations are given equal weight. The work here generalises their result to arbitrary λ and this is believed to be the first example of a soluble interacting walk model with more than two walks having a variable interaction parameter. In ref. 17 the method of differential approximants reviewed in ref. 19 was used to determine a recurrence relation, having polynomial coefficients, satisfied by the sequence of partition functions for increasing t. Although the recurrence relation was not proven, the values of t used made it inconceivable that it would ever fail. For general λ the same method leads to a recurrence relation of order 5 with coefficients linear in t from which the generating function may be deduced by solving a first order differential equation. However in Section 3 a different approach is used which gives a much neater recurrence relation with constant coefficients and an inhomgeneous term which is the product of an exponential and a linear combination of Catalan numbers C_t

$$2(\lambda-1)(\lambda-2)\, Z_t^{(3)} - 2\lambda(\lambda^2+3\lambda-8)\, Z_{t-1}^{(3)} + 16\lambda^2 Z_{t-2}^{(3)} + 8\lambda^4 Z_{t-3}^{(3)} = 2^t R_t \qquad (4)$$

where $R_t=(\lambda-4)(\lambda^2 C_{t-2}-4C_t)$. The relation becomes homogeneous at the collapse point $\lambda=4$ at which point the simple solution $Z_t^{(3)}=8^t$ also satisfies the initial conditions. The form is motivated by the corresponding relation for two walkers (Section 2) which we are able to derive from the known generating function.[18] It is hoped that a similar approach may work for more than three walkers if the appropriate form of the right-hand side can be found. Also the relative simplicity of the relation gives rise to the hope that a proof may be possible.

In Section 4 the asymptotic form of the partition function is obtained in the limit $t \to \infty$. It is found that

$$Z_t^{(n)}(\lambda) \sim [\mu^{(n)}(\lambda)]^t \, t^g \tag{5}$$

where the growth factor $\mu^{(n)}(\lambda)$ varies continuously with λ. $G^{(n)}(\lambda, u)$, as a function of u, will have at least one singular point on the positive real axis. The closest such point will be at $u_c^{(n)}(\lambda) = 1/\mu^{(n)}(\lambda)$ and as $u \to u_c^{(n)}(\lambda)$ from below we find that, for $\lambda \neq \lambda_c$,

$$G^{(n)}(\lambda, u) \sim |u - u_c^{(n)}(\lambda)|^{-\gamma} \tag{6}$$

where $\gamma = g + 1$.

Our results for the growth factor and exponents are summarised in Table I. The collapse point is found to be same for two and three walks, that is $\lambda_c = 4$. This may be understood by considering the case when two walkers arrive at the same site. In the case of osculating walkers there is only one way for them to leave but if the walks were allowed to share the same bond and to cross there would be 4 ways, so placing weight 4 on an osculating vertex is equivalent to allowing the walks to move independently. Thus for any number of walks $\lambda_c = 4$, $Z_t^{(n)}(4) = 2^{nt}$ and the value of g is therefore zero. For $\lambda > \lambda_c$, $G^{(n)}(\lambda, u)$ has several singular points on

Table I. Growth Factors and Exponents. The Confluent Exponents at $\lambda = \lambda_c = 4$ Are Not Present for $\varepsilon = 1$

	Two walks			Three walks		
	μ	g	g_ε	μ	g	g_ε
$\lambda < 4$	4	$-\frac{1}{2}$	$-\frac{3}{2}$	8	$-\frac{3}{2}$	$-\frac{5}{2}$
$\lambda = 4$	4	$0, -\frac{1}{2}$	$-\frac{1}{2}$	8	$0, -\frac{1}{2}$	$-\frac{1}{2}$
$\lambda > 4$	$\frac{\lambda}{\sqrt{\lambda}-1}$	0	0	$\frac{\lambda^2}{\lambda-2}$	0	0

the real axis the positions of which depend on λ. The collapse transition is marked by a subset of these coming together at $u = 1/2^n$ as $\lambda \to \lambda_c$ at which point cancellation takes place leaving the simple pole $G^{(n)}(4, u) = 1/(1 - 2^n u)$.

For $\lambda < 4$ the growth factor $\mu^{(n)}(\lambda) = 2^n$ and the exponent g is the same as for vicious walker star configurations, that is from Fisher's formula (1)

$$g = g_S = -\frac{n(n-1)}{4} \tag{7}$$

These results are expected to be true for any number of walks.

For $\lambda > 4$ the walks tend to stick together and for both two and three walks the exponent is the same as that for a single walk (i.e., $g = 0$). In this region the growth factor varies with λ and as $\lambda \to \infty$, $\mu^{(n)}(\lambda) \sim \lambda^{\frac{n-1}{2}}$ which is the expected form when the osculating walks are completely bound together in a single rod-like configuration. In this configuration there is one osculation every two steps in the case of two walks but one on every step for three walks which explains the power of λ. Again these results are expected to be valid for any number of walks.

The model may be further generalised by including a weight factor ε whenever two walks terminate on the same site. The coefficient $z_{t,i}^{(n)}$ is now replaced by $z_{t,i}^{(n)}(\epsilon) = \sum_{f=0}^{\lfloor n/2 \rfloor} z_{t,i,f}^{(n)} \varepsilon^f$ where $z_{t,i,f}^{(n)}$ is the number of n-walk configurations having t steps and i osculations which make f intersections on the final step. For $\lambda = 0$ and $\epsilon = 0$ the partition function is exactly equal to the number vicious walker star configurations whereas for the previous model ($\epsilon = 1$) the equality when $\lambda = 0$ was only asymptotic. For two or three chains $f = 0$ or 1, since only one walk can traverse a given bond, so $G^{(n)}(\lambda, u)$ is linear in ε and we write $G^{(n)}(\lambda, u) = G^{(n)}(\lambda, u)|_{\varepsilon = 1} + (\varepsilon - 1)\, G_\varepsilon^{(n)}(\lambda, u)$. In the case of two chains $G_\varepsilon^{(n)}(\lambda, u)$ is the partition function for watermelon configurations and for three chains it is the partition function for networks in which the endpoints of just two of the chains are joined together. Such networks have been called "ceratic."[20] For general ε it is found that (2) still holds but the inhomogeneous part is replaced by $R_t(\varepsilon) = (1 - \varepsilon)(4C_{t+1} - \lambda^2 C_{t-1}) - (\lambda - 4\varepsilon)(4C_t - \lambda^2 C_{t-2})$. It is interesting that the linear part of the recurrence relation is independent of ε.

For $\lambda < 4$, $G_\varepsilon^{(n)}(\lambda, u)$ has exponent $\gamma_\varepsilon = g_\varepsilon + 1$ where for two walks $g_\varepsilon = -\frac{3}{2}$, the exponent for staircase polygons.[21] For three walks $g_\varepsilon = -\frac{5}{2}$ is the exponent for the ceratic network and is in agreement with the exponent formula[4] for general fully directed polymer networks corresponding to vicious walker configurations. The contribution of $G_\varepsilon^{(n)}(\lambda, u)$ to $G^{(n)}(\lambda, u)$ is of course asymptotically negligible and merges with the correction terms.

For $\varepsilon \neq 1$ and $\lambda > 4$ the asymptotic form is still dominated by a simple pole the position of which is independent of ε. Also for $\varepsilon \neq 1$, $R_t(\varepsilon)$ no longer vanishes at $\lambda = 4$ and the t dependence of $Z_t^{(n)}(4)$ is not just a simple exponential. When the $\lambda > 4$ singularities of $G^{(n)}(\lambda, u)$ merge as $\lambda \to 4$ the simple pole which is left behind for $\varepsilon = 1$ is augmented by the term $(\varepsilon - 1)\, G_\varepsilon^{(n)}(4, u)$. This gives rise to an important confluent singularity which has exponent $\gamma_\varepsilon = g_\varepsilon + 1$ where for two walks $g_\varepsilon = -\frac{1}{2}$, the exponent for walks which are independent except that they must intersect after t-steps.[7, 1] This exponent is the same for both two and three walk configurations because in the latter case only two of the walks have a common endpoint and the third walk is completely independent.

In Section 5 the formula for $G^{(n)}(\lambda, u)$ is used to obtain $z_{t,i}^{(n)}(\epsilon)$, the micro canonical partition function. It is found that for two and three walkers $z_{t,i}^{(n)}(\epsilon)$ may be expressed as a linear combination of Ballot numbers. This could also be the case for $n \geqslant 4$.

2. TWO WALKS

The two walker problem may be solved exactly in various ways[1, 17, 18] and we present results for this case as a guide to the solution of the three walk problem. It is also of interest to compare the formulae for the two problems since they have some common features.

First consider t-step configurations with no osculations and denote the generating function by $G_0^{(2)}(u)$.

- Configurations in which the walks terminate on the same site are equinumerous with staircase polygons of length $t+1$ the number of which is known[21] to be the Catalan number

$$C_t = \frac{1}{t+1} \binom{2t}{t} \tag{8}$$

having generating function

$$c^*(u) = \sum_{t=1}^{\infty} C_t u^t = \frac{1 - 2u - \sqrt{1-4u}}{2u} \tag{9}$$

which satisfies the equation

$$uc^{*2} - (1-2u)\, c^* + u = 0. \tag{10}$$

- Configurations in which the walks terminate on different sites biject to pairs of $(t+1)$-step walks which start together and never meet again.

The generating function for such walks was shown in ref. 7 (Eq. (14)) to be $(1-4u)^{-\frac{1}{2}}$ but includes the zero length configuration which must be subtracted and configurations with $t>1$ were counted twice because the walks were considered distinguishable.

To keep account of the two types of configuration we give weight ε to configurations in which the walks end on the same site thus, using (9)

$$G_0^{(2)}(u)=\varepsilon c^*(u)+\frac{1}{2u}\left(\frac{1}{\sqrt{1-4u}}-1\right)=\varepsilon c^*(u)+\frac{1-c^*(u)}{1-4u}. \tag{11}$$

Following Fisher[1] we find the generating function $G^{(2)}(\lambda, u)$ using the bubble chaining technique. Any configuration with i osculations may be obtained by concatenating i staircase polygons, moving the first pair of steps to the end and appending a configuration with no osculations. The generating function for configurations with i osculations is therefore

$$G_i^{(2)}(u)=(uc^*(u))^i\, G_0^{(2)}(u) \tag{12}$$

and hence

$$G^{(2)}(\lambda, u)=\sum_{i=0}^{\infty} G_i^{(2)}(u)\,\lambda^i=\frac{G_0^{(2)}(u)}{1-\lambda uc^*(u)}. \tag{13}$$

To obtain a recurrence relation for $Z_t^{(2)}(\lambda)$ we note that, using (10)

$$\frac{1}{1-\lambda uc^*(u)}=\frac{\lambda-1-2\lambda u-\lambda uc^*(u)}{\lambda-1-2\lambda u-\lambda^2u^2} \tag{14}$$

and combining this with (13) and (11) and using (10) gives

$$G^{(2)}(\lambda, u)=\frac{\lambda-1-3\lambda u+\lambda\varepsilon u(1-4u)+(1-\lambda u-\varepsilon(1-4u))\,c^*(u)}{(1-4u)(\lambda-1-2\lambda u-\lambda^2u^2)} \tag{15}$$

and hence, for $t\geqslant 3$,

$$\begin{aligned}(\lambda-1)\,Z_t^{(2)}-2(3\lambda-2)\,Z_{t-1}^{(2)}-\lambda(\lambda-8)\,Z_{t-2}^{(2)}+4\lambda^2Z_{t-3}^{(2)}\\ =(1-\varepsilon)\,C_t-(\lambda-4\varepsilon)\,C_{t-1}.\end{aligned} \tag{16}$$

For the case of vicious walkers, $\lambda=0$, the relation is valid for $t\geqslant 2$. In order to find the generating function for three walk configurations we will first look for a recurrence relation of a form similar to that of (16).

3. THREE WALKS

In the case of three walks the bubble chaining technique is insufficient to give an explicit formula and the results of the following sections are obtained by fitting a recurrence relation to the sequence of partition functions for increasing values of t. The partition functions are easily generated using the partial difference equations

$$\begin{aligned}
Z_0(x, x, x_3) &= Z_0(x_1, x, x) = \varepsilon \\
Z_0(x_1, x_2, x_3) &= 1 \\
Z_t(x, x, x_3) &= \lambda Z_{t-1}(x-1, x+1, x_3-1) + \lambda Z_{t-1}(x-1, x+1, x_3+1) \\
Z_t(x_1, x, x) &= \lambda Z_{t-1}(x_1-1, x-1, x+1) + \lambda Z_{t-1}(x_1+1, x-1, x+1) \\
Z_t(x_1, x_2, x_3) &= \sum_{\delta_1=\pm 1} \sum_{\delta_2=\pm 1} \sum_{\delta_2=\pm 1} Z_{t-1}(x_1+\delta_1, x_2+\delta_2, x_3+\delta_3) \\
Z_t^{(3)}(\lambda) &= Z_t(0, 2, 4)
\end{aligned} \tag{17}$$

where $Z_t(x_1, x_2, x_3)$ is the partition function for walks of length t which start at positions $x_1 \leqslant x_2 \leqslant x_3$. The first few partition functions are as follows. $Z_t^{(3)}(\lambda) = Z_{t,1}^{(3)}(\lambda) + (\epsilon - 1)\, Z_{t,\epsilon}^{(3)}(\lambda)$ where

$$\begin{aligned}
&Z_{0,1}^{(3)}(\lambda) = 1 \qquad Z_{1,1}^{(3)}(\lambda) = 8 \\
&Z_{2,1}^{(3)}(\lambda) = 32 + 8\lambda \\
&Z_{3,1}^{(3)}(\lambda) = 160 + 72\lambda + 4\lambda^2 \\
&Z_{4,1}^{(3)}(\lambda) = 896 + 480\lambda + 64\lambda^2 + 4\lambda^3 \\
&Z_{5,1}^{(3)}(\lambda) = 5376 + 3136\lambda + 640\lambda^2 + 56\lambda^3 + 4\lambda^4 \\
&Z_{6,1}^{(3)}(\lambda) = 33792 + 20736\lambda + 5248\lambda^2 + 640\lambda^3 + 64\lambda^4 + 4\lambda^5 \\
&Z_{7,1}^{(3)}(\lambda) = 219648 + 139392\lambda + 40128\lambda^2 + 6304\lambda^3 + 720\lambda^4 + 72\lambda^5 + 4\lambda^6
\end{aligned}$$

and

$$\begin{aligned}
&Z_{0,\epsilon}^{(3)}(\lambda) = 0 \qquad Z_{1,\epsilon}^{(3)}(\lambda) = 4 \\
&Z_{2,\epsilon}^{(3)}(\lambda) = 12 + 2\lambda \\
&Z_{3,\epsilon}^{(3)}(\lambda) = 48 + 24\lambda + 2\lambda^2 \\
&Z_{4,\epsilon}^{(3)}(\lambda) = 224 + 144\lambda + 20\lambda^2 + 2\lambda^3 \\
&Z_{5,\epsilon}^{(3)}(\lambda) = 1152 + 832\lambda + 192\lambda^2 + 24\lambda^3 + 2\lambda^4 \\
&Z_{6,\epsilon}^{(3)}(\lambda) = 6336 + 4896\lambda + 1456\lambda^2 + 216\lambda^3 + 28\lambda^4 + 2\lambda^5 \\
&Z_{7,\epsilon}^{(3)}(\lambda) = 36608 + 29568\lambda + 10208\lambda^2 + 1888\lambda^3 + 280\lambda^4 + 32\lambda^5 + 2\lambda^6
\end{aligned}$$

Each sequence was fitted to a recurrence relation of the form

$$\sum_{r=0}^{d} a_r(\lambda)\, Z_{t-r} = 2^t \sum_{n=n_1}^{n_2} b_n(\lambda)\, C_{t-n}$$

by trying values of the parameters d, n_1 and n_2. Sufficient terms of the sequence were used to solve for the coefficients $a_r(\lambda)$ and $b_n(\lambda)$ (using Mathematica Solve) and the resulting recurrence relation was used to compare with the unused terms available. For both sequences a fit was found with $d=3$, $n_1=-1$ and $n_2=2$. Since the partition function is linear in ϵ this was sufficient to obtain the ϵ dependence with the result

$$2(\lambda-1)(\lambda-2)\, Z_t^{(3)} - 2\lambda(\lambda^2+3\lambda-8)\, Z_{t-1}^{(3)} + 16\lambda^2 Z_{t-2}^{(3)} + 8\lambda^4 Z_{t-3}^{(3)} = 2^t R_t \tag{18}$$

where

$$R_t = (1-\varepsilon)(4C_{t+1} - \lambda^2 C_{t-1}) - (\lambda - 4\varepsilon)(4C_t - \lambda^2 C_{t-2}). \tag{19}$$

This relation has not been proven but is almost certainly exact since far more terms were generated than were required to determine the coefficients and these were in agreement with the relation.

The chosen form of the recurrence relation was suggested by that for two walkers but it was found necessary to include a factor 2^t on the right-hand side multiplying the linear combination of Catalan numbers. Such a factor occurs in the number of vicious three-walker star configurations (the case $\lambda=\epsilon=0$) which is known to be $2^t C_{t+1}$.[11, 13] Also when $\lambda=0$ the coefficient of ϵ in $Z_t^{(3)}(\lambda)$ is the number of vicious three-walker configurations, two walks of which move to the same site on the last step. Using the correspondence with Young tableaux[9] it may be shown, using a Pieri formula,[22] that this number is $2^t(4C_t - C_{t+1})$. Both of these results follow from our conjectured recurrence relation. A rationale for the factor 2^t is the fact that the walks only interact in pairs so that, at any step, one of the walks can move in either direction without interacting.

Let $Z_t = 2^t Y_t$ then

$$2(\lambda-1)(\lambda-2)\, Y_t - \lambda(\lambda^2+3\lambda-8)\, Y_{t-1} + 4\lambda^2 Y_{t-2} + \lambda^4 Y_{t-3} = R_t. \tag{20}$$

and with the initial conditions $Y_0=1$, $Y_1=2(1+\varepsilon)$ and $Y_2 = 5+\frac{3}{2}\lambda + 3\varepsilon + \frac{1}{2}\varepsilon\lambda$ the generating function is found to be $G^{(3)}(\lambda, u) = Y(\lambda, 2u)$ where

$$Y(\lambda, u) = \frac{u f(\lambda, u) + (4 - u^2\lambda^2)(1 - \lambda u - \varepsilon(1-4u))\, c^*(u)}{u(2(\lambda-2) - u\lambda^2)(\lambda - 1 - 2u\lambda - u^2\lambda^2)} \tag{21}$$

and

$$f(\lambda, u) = 2(\lambda(\lambda-3)+2\varepsilon) - \lambda u(\lambda(\lambda-1)-4(\lambda-3)\,\varepsilon) + \lambda^2 u^2(\lambda\,(1-\varepsilon)-4\varepsilon) \tag{22}$$

Notice that one of the quadratic denominator factors of (21) is the same as that for two walkers and we may therefore use (14) to replace it by a factor which is linear in λ. A similar replacement is possible for the other factor and we find

$$Y(\lambda, u) = \frac{\varepsilon u(2+\lambda u) + (2-\lambda u - \varepsilon(2-8u+\lambda u-\lambda u^2))\, c^*(u)}{2u(1-\lambda u c^*(u))(1-\lambda u(1+c^*(u))/2)} \tag{23}$$

where the numerator has also been reduced in degree from cubic to linear.

Setting $\lambda = 1$ in (21) gives the total number of osculating configurations

$$G^{(3)}(1, u) = \frac{2u(u-1+\varepsilon(1-5u)) + (1-u)(1-2u-\varepsilon(1-8u))\, c^*(2u)}{4u^2(1+u)} \tag{24}$$

$$= \frac{\phi(u) - (1-u)(1-2u-\varepsilon(1-8u))\sqrt{1-8u}}{16u^3(1+u)} \tag{25}$$

where $\phi(u) = (1-u)(1-6u) - \varepsilon(1-13u+36u^2+8u^3)$. This agrees with the result of ref. 17 (Eq. (4.36)) when $\varepsilon = 1$.

4. CRITICAL BEHAVIOUR

(a) $\lambda < 4$

Notice that $c^*(u)$ increases monotonically from 0 to 1 as u goes from 0 to the singular point, $u = \frac{1}{4}$, of $c^*(u)$ so that in this region of λ and u the denominators of both two and three walk functions are strictly positive. As $u \to \frac{1}{4}$ from below

$$G^{(2)}(\lambda, u) = \frac{4}{4-\lambda}\left[\frac{2}{\sqrt{1-4u}} + \varepsilon - 2\,\frac{4+\lambda}{4-\lambda} + O((1-4u)^{\frac{1}{2}})\right] \tag{26}$$

$$G^{(3)}(\lambda, u) = \frac{4}{(4-\lambda)^2}\left[2(8-\lambda) + \varepsilon(4-\lambda) - \frac{2(64-\lambda^2)\sqrt{1-8u}}{4-\lambda} + O(1-8u)\right]. \tag{27}$$

$$G_\varepsilon^{(2)}(\lambda, u) = \frac{4}{4-\lambda}\left[1 - \frac{8\sqrt{1-4u}}{4-\lambda} + O(1-4u)\right] \tag{28}$$

$$G_\varepsilon^{(3)}(\lambda, u) = \frac{4}{4-\lambda}\,[1-(64-4\lambda-\lambda^2)\, v^2 + 8(64-\lambda^2)\, v^3 + O(v^4)] \tag{29}$$

where $v = (1-8u)^{\frac{1}{2}}/(4-\lambda)$.

(b) $\lambda > 4$

If and only if $\lambda > 4$, the denominator of $G^{(2)}(\lambda, u)$ has a simple zero at

$$u_c^{(2)}(\lambda) = 1/\mu^{(2)}(\lambda) = \frac{\sqrt{\lambda}-1}{\lambda} < \frac{1}{4} \tag{30}$$

and therefore the dominant singularity of $G^{(2)}(\lambda, u)$ is a simple pole at this position.

Notice that $Y(u)$ also has a pole at this position which means that there is a pole in $G^{(3)}(\lambda, u)$ at $\frac{1}{2}u_c^{(2)}(\lambda)$. However there is a second pole which is closer to the origin arising from the other denominator in (23). The dominant singularity of $G^{(3)}(\lambda, u)$ is therefore a pole at

$$u_c^{(3)}(\lambda) = 1/\mu^{(3)}(\lambda) = \frac{\lambda-2}{\lambda^2} \tag{31}$$

(c) $\lambda=4$

Setting $\lambda = 4$ in (15) gives the generating function for two walks at the collapse transition as

$$G^{(2)}(4, u) = \frac{1}{1-4u} + (\varepsilon-1)\, G_\varepsilon^{(2)}(4, u) \tag{32}$$

where

$$G_\varepsilon^{(2)}(4, u) = \frac{(4u-c^*(u))}{(1-4u)(3+4u)} = \frac{1}{2u(3+4u)}\left[\frac{1}{\sqrt{1-4u}} - 1 - 2u\right] \tag{33}$$

Notice that when $\varepsilon = 1$ the generating function is that for independent walkers. This is the case for any number of walkers as explained earlier. When $\varepsilon \neq 1$ there is a confluent singularity with $\gamma = \frac{1}{2}$ which is the dominant singularity of $G_\varepsilon^{(2)}(4, u)$.

Similar behaviour is found for three walkers except that $u_c = \frac{1}{8}$ instead of $\frac{1}{4}$.

$$G^{(3)}(4, u) = \frac{1}{1-8u} + (\varepsilon - 1)\, G_\varepsilon^{(3)}(4, u) \tag{34}$$

where

$$G_\varepsilon^{(3)}(4, u) = \frac{(2u(1+16u) - (1-16u^2)\, c^*(2u))}{2u(1-8u)(3+8u)} \tag{35}$$

$$= \frac{1}{8u^2(3+8u)} \left[\frac{1-16u^2}{\sqrt{1-8u}} - 1 - 4u - 8u^2 \right] \tag{36}$$

5. THE MICRO CANONICAL ENSEMBLE

The micro canonical partition function $z_{t,i}(\epsilon)$ is a sum over all t-step configurations having i osculations and weighted by ϵ^f where f is the number of intersections on the final step. The main interest here is combinatorial.

In the case of two and three walks we obtain explicit expressions for $z_{t,i}(\epsilon)$ as linear combinations of Ballot numbers. For three walks this is probably the first such expression of its kind and comparison of the two and three walk formulae may give a clue as to how tackle the problem of more than three walkers. Since in this case $f = 0$ or 1 the expressions are linear in ϵ. For $n = 2$ and 3, the canonical partition function $Z_t^{(n)}(\lambda, \epsilon) \equiv \sum_{i=0}^{\lfloor \frac{(n-1)t}{2} \rfloor} z_{t,i}^{(n)} \lambda^i \varepsilon^f$ is also a linear combination of Ballot numbers the coefficients of which are polynomials in λ but no obvious simplication arises on including the additional summation.

5.1. Two Walks

Now[23]

$$c^*(u)^y = \sum_{t=y}^{\infty} B_{2t-1,\, 2y-1} u^t \tag{37}$$

where $B_{-1,-1} = 1$; for $j \neq 1$, $B_{-1,j} = B_{j,-1} = 0$ and for $j, k \geqslant 0$, $B_{j,k}$ is the Ballot number

$$B_{j,k} = \frac{(k+1)\, j!}{(\frac{1}{2}(j+k)+1)!\, (\frac{1}{2}(j-k))!} \tag{38}$$

Expanding (12) gives the number of configurations with i osculations as

$$z^{(2)}_{t,i}(\epsilon) = \varepsilon B_{2t-2i-1,\,2i+1} + \sum_{\ell=0}^{t-2i} 4^{\ell}(B_{2t-2i-2\ell-1,\,2i-1} - B_{2t-2i-2\ell-1,\,2i+1}) \tag{39}$$

for $t \geqslant 2i$ and zero otherwise.

5.2. Three Walks

To obtain $z^{(3)}_{t,i}(\epsilon)$ from (23) requires the expansion of $1/(1-\lambda uc^*(u))$ and $1/(1-\lambda u(1+c^*(u))/2)$ in powers of λ and then a further expansion of the resulting functions $c^*(u)^y$ and $(1+c^*(u))^y$ in powers of u. The first of these u expansions is the same as for the two walk problem and is given by (37) and the second is[20]

$$(1+c^*(u))^y = \sum_{s=0}^{\infty} B_{2s+y-1,\,y-1} u^s \tag{40}$$

Expanding (23) directly in powers of λ and u would therefore give an expression for $z^{(3)}_{t,i}(\epsilon)$ involving products of Ballot numbers. However first rewriting $Y(\lambda, u)$ in the form

$$Y(\lambda, u) = Y_1(\lambda, u) + Y_2(\lambda, u) \tag{41}$$

where

$$Y_1(\lambda, u) = \frac{2(1-\varepsilon) + 8\varepsilon u - (1+\varepsilon - 3\varepsilon u)\, u\lambda + \varepsilon u^3\lambda^2}{u^2\lambda(1-\lambda u)(1-\lambda uc^*(u))} \tag{42}$$

and

$$Y_2(\lambda, u) = \frac{-2(1-\varepsilon) - 8\varepsilon u + 2(1+\varepsilon\, u)\, u\lambda - \frac{1}{2}(1+\varepsilon)\, u^2\lambda^2}{u^2\lambda(1-\lambda u)(1-\lambda u(1+c^*(u))/2)} \tag{43}$$

produces a linear combination of Ballot numbers. Thus defining

$$X_{i,1}(u) \equiv u^i \sum_{j=0}^{i} c^*(u)^j \qquad \text{and} \qquad X_{i,2}(u) \equiv u^i \sum_{j=0}^{i} 2^{-j}(1+c^*(u))^j \tag{44}$$

the generating function for configurations with exactly i osculations is then

$$G^{(3)}_i(u) = Y_{i,1}(2u) + Y_{i,2}(2u) \tag{45}$$

where

$$Y_{i,1}(u) = \frac{2(1-\varepsilon)+8\varepsilon u}{u^2} X_{i+1,1}(u) - \frac{1+\varepsilon-3\varepsilon u}{u} X_{i,1}(u) + \varepsilon u X_{i-1,1}(u) \tag{46}$$

and

$$Y_{i,2}(u) = \frac{-2(1-\varepsilon)-8\varepsilon u}{u^2} X_{i+1,2}(u) + \frac{2(1+\varepsilon u)}{u} X_{i,2}(u) - \frac{1+\varepsilon}{2} X_{i-1,2}(u) \tag{47}$$

and further defining

$$b_{t,i,1} \equiv \sum_{j=0}^{i} B_{2t-2i-1,2j-1} \qquad \text{and} \qquad b_{t,i,2} \equiv \sum_{j=0}^{i} 2^{-j} B_{2t-2i+j-1,j-1} \tag{48}$$

gives the number of t-step configurations with i osculations as

$$z_{t,i}^{(3)}(\epsilon) = 2^t (y_{t,i,1} + y_{t,i,2}) \tag{49}$$

where

$$y_{t,i,1} = 2(1-\varepsilon)\, b_{t+2,i+1,1} + 8\varepsilon\, b_{t+1,i+1,1} - (1+\varepsilon)\, b_{t+1,i,1} + 3\varepsilon\, b_{t,i,1} + \varepsilon\, b_{t-1,i-1,1} \tag{50}$$

and

$$y_{t,i,2} = -2(1-\varepsilon)\, b_{t+2,i+1,2} - 8\varepsilon\, b_{t+1,i+1,2} + 2b_{t+1,i,2} + 2\varepsilon\, b_{t,i,2} - \frac{1+\varepsilon}{2} b_{t,i-1,2} \tag{51}$$

When $i = 0$ this collapses to the simpler formula for vicious walker configurations

$$z_{t,0}^{(3)}(\epsilon) = 2^t C_{t+1} + \varepsilon\, 2^t (4C_t - C_{t+1}) \tag{52}$$

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